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THE PROBLEM OF A PISTON IN STRATIFIED GAS WITH WEAKLY CHANGING PARAMETERS*

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The problem of a piston in gas whose initial parameters (density, temperature, and the adiabatic exponent) may vary along the normal to the direction of piston motion is considered.

The uneven distribution of initial parameters in a medium can be due to a number of factors. For instance, the effect of gravity leads to the stratification of gas by density and temperature. Stratification by density and the adiabatic exponent can be due to the presence in the gas of small solid particles. Indeed, if the nonequilibrium state of the gas and particle mixture is disregarded, assuming the temperatures and velocities of these to be the same /1-3/, the system of equations defining the flow of such medium is a set of equations of motion and mass and energy balance of some perfect gas with reduced physical parameters, viz. density $\rho = \rho_g + \rho_s$ and the adiabatic exponent $k = (c_p + \varkappa c_s)(c_v + \varkappa c_s)^{-1}$, where c_p and c_v are specific heats of the gas phase, and c_s is the specific heat of particles; ρ_g and ρ_s are, respectively, the densities of gas and particles. The quantity $\varkappa = \rho_s/\rho_g$ is constant in particle and continuous at the shock wave front. The nonuniform distribution of particles in the gas leads to stratification by density and the adiabatic exponent. This case is investigated below in connection with the problem of shock waves propagation in coal mines. Other variants of initial stratification can be dealt with in a similar manner.

1. A flat piston whose plane lies in the YZ plane moves along the OX axis at constant velocity C in the quiescent gas with an admixture of solid particles. This mixture is defined by the initial pressure p_0, constant density of the gas phase ρ_{g0}, and variable solid phase density $\rho_{s0} = \rho_{s0}^* + \varepsilon\rho_{g0}\varphi(\delta Y)$, where $1/\delta$ is some characteristic linear dimension, and ρ_{s0}^* is a constant. The gas phase is assumed to be a perfect gas with the adiabatic exponent γ. The piston motion through the medium induces the propagation of a curved shock wave, with a unsteady two-dimensional flow in the region between the shock wave and piston. We shall solve the problem in the single-fluid approximation. Then, as shown above, the flow of mixture is defined by two dimensional unsteady equations of adiabatic motion of gas, whose adiabatic exponent depends on the flow parameters. We also assume that $\varepsilon \ll 1$, and shall solve the problem in linear approximation /4-6/.

The flow of gas induced by the piston motion in a homogeneous medium (with constant initial density, $\varepsilon = 0$) is defined, as known, by formulas

$$\frac{\rho_1}{\rho_0^*} = \frac{k_0+1}{k_0-1}\left[1 + \frac{2}{M_0^2(k_0+1)}\right]^{-1}, \quad \frac{p_1}{p_0} = \frac{1-k_0}{1+k_0}\left(1 - \frac{2k_0M_0^2}{k_0-1}\right)$$

$$\frac{C+D}{D} = \frac{\rho_1}{\rho_0^*}$$

$$\rho_0^* = \rho_{s0} + \rho_{g0}, \quad M_0 = \frac{C+D}{a_0}, \quad k_0 = \frac{c_p + \varkappa_0 c_s}{c_v + \varkappa_0 c_s}, \quad \varkappa_0 = \frac{\rho_{s0}}{\rho_{g0}}$$

where D is the shock wave velocity relative to the piston, ρ_0^* is the gas density, a_0 is the speed of sound, M_0 is the Mach number ahead of the shock wave front, ρ_1, p_1 are, respectively, the density and pressure of gas behind the shock, and k_0 is the adiabatic exponent.

Owing to the nonuniform particle distribution, perturbations of pressure p', density ρ' and of velocity components u' and v' propagate on this zero background.

We introduce dimensionless coordinates, time and the dimensionless unknown functions

$$x = \delta X, \quad y = \delta Y, \quad \tau = \delta a_1 t, \qquad u = \frac{u'}{C}, \quad v = \frac{v'}{C}, \quad \omega = \frac{p'}{\rho_1 a_1 C}$$

*Prikl.Matem.Mekhan.,46,No.3,pp.429-434,1982 where a_1 is the speed of sound behind the shock wave front in the unperturbed stream. The linearized equations for perturbations can now be reduced to the form

$$\frac{\partial\omega}{\partial\tau}+\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \qquad (1.1)$$
$$\frac{\partial u}{\partial\tau}+\frac{\partial\omega}{\partial x}=0,\qquad \frac{\partial v}{\partial\tau}+\frac{\partial\omega}{\partial y}=0$$

The conditions at the shock wave front in dimensionless form are

$$u=\frac{1+\sigma}{2\beta}g_\tau+(\sigma+\alpha)\psi(y),\qquad v=-\frac{1}{2\lambda h}g_y \qquad (1.2)$$
$$\omega=g_\tau+\beta(1+\alpha)\psi(y)\quad \text{at}\quad x=\beta\tau$$

where

$$h=\frac{2}{k_0+1},\qquad \sigma=\frac{1}{M_0^2},\qquad \lambda=\frac{D}{C},\qquad \beta=\frac{D}{a_1}$$
$$\alpha=\frac{hk_1}{2k_0}(1+\varkappa_0)(k_0+\sigma),\qquad k_1=\frac{c_s(c_p-c_v)}{(c_v+\varkappa_0 c_s)^2}$$
$$\psi(y)=\frac{h(1+\lambda)}{1+\varkappa_0}\varphi(y),\qquad g(\tau,y)=2\lambda h\delta f(t,Y)$$

and $f(t,Y)$ is the deviation of the shock wave front from the straight line. At the piston surface $x=0$ the condition $u=0$ is satisfied.

Differentiation of the last of formulas (1.2) along the line $x=\beta\tau$ with (1.1) taken into account enables us to eliminate from (1.2) $g(\tau,y)$, and write the boundary conditions (1.2) as

$$\frac{\partial v}{\partial x}=A\frac{\partial\omega}{\partial y}+a\psi',\qquad u=B\omega+b\psi\quad \text{at}\quad x=\beta\tau \qquad (1.3)$$
$$A=\frac{1}{\beta}\left(1-\frac{1}{2\lambda h}\right),\qquad B=\frac{1+\sigma}{2\beta},\qquad a=\frac{1+\alpha}{2\lambda h}$$
$$b=\frac{(\sigma-1)(1-\alpha)}{2}$$

At the initial instant of piston motion $\tau=0$ the shock wave front coincided with the piston face, i.e. $g(0,y)=0$. Then from the last formula of (1.2) follows that $v=0$ when $\tau=0$, and from the last formula of (1.3) we have $\omega=-bB^{-1}\psi$. Hence we must supplement boundary conditions (1.3) with the initial conditions

$$u=v=0,\quad \omega=\omega_0\psi(y),\quad \omega_0=-bB^{-1}\quad \text{at}\quad x=\tau=0 \qquad (1.4)$$

As the result, we have to solve the hyperbolic system of Eq.(1.1) in the region $0\leqslant x\leqslant\beta\tau$, $-\infty<y<+\infty$, $\tau\geqslant 0$ with boundary and initial conditions (1.3) and (1.4), and the condition

$$u=0\quad \text{at}\quad x=0 \qquad (1.5)$$

2. In conformity with /6/ we carry out the coordinate transformation

$$x=r\,\mathrm{sh}\,\theta,\quad \tau=r\,\mathrm{ch}\,\theta,\quad y=y \qquad (2.1)$$

and after such substitution and elementary transformations of system (1.1) obtain the system of equations

$$\frac{\partial\omega}{\partial r}+\frac{1}{r}\frac{\partial u}{\partial\theta}+\mathrm{ch}\,\theta\frac{\partial v}{\partial y}=0,\qquad \frac{\partial u}{\partial r}+\frac{1}{r}\frac{\partial\omega}{\partial\theta}+\mathrm{sh}\,\theta\frac{\partial v}{\partial y}=0 \qquad (2.2)$$
$$\mathrm{ch}\,\theta\frac{\partial v}{\partial r}-\frac{\mathrm{sh}\,\theta}{r}\frac{\partial v}{\partial\theta}+\frac{\partial\omega}{\partial y}=0$$

Condition (1.3), (1.4), and (1.5) can be similarly reduced to the form

$$\frac{\partial v}{\partial r}=(A\,\mathrm{sh}\,\theta_0-\mathrm{ch}\,\theta_0)\frac{\partial\omega}{\partial y}+a\,\mathrm{sh}\,\theta_0\,\psi',\quad u=B\omega+b\psi\quad \text{when}\quad \theta=\theta_0 \qquad (2.3)$$
$$u=0\ \text{when}\ \theta=0;\quad u=v=0,\quad \omega=\omega_0\psi(y)\ \text{when}\ r=0$$

As the result of coordinate transformation (2.1) the plane $x=0$ becomes the plane $\theta=0$, and plane $x=\beta\tau$ becomes plane $\theta=\theta_0$, while $\mathrm{th}\,\theta_0=\beta$.

Solution of the system of Eqs.(2.2) with conditions (2.3) can be obtained in the form of series

$$\omega = \omega_0\psi + \sum_{k=1}^{\infty} \omega_{2k}(\theta)\,\psi^{(2k)} r^{2k}, \qquad u = \sum_{k=1}^{\infty} u_{2k}(\theta)\,\psi^{(2k)} r^{2k} \tag{2.4}$$
$$v = [-\omega_0 \operatorname{ch}\theta + (A\omega_0 + a)\operatorname{sh}\theta]\,\psi' r + \sum_{k=1}^{\infty} v_{2k+1}(\theta)\,\psi^{(2k+1)} r^{2k+1}$$

where ω_k, u_k, v_k are functions of the single variable θ that satisfy the recurrent formulas

$$\omega_k = \frac{Q_k \operatorname{ch} k\theta - p_k B \operatorname{sh} k(\theta_0 - \theta) - p_k \operatorname{ch} k(\theta_0 - \theta)}{B \operatorname{ch} k\theta_0 + \operatorname{sh} k\theta_0} -$$
$$\frac{1}{k}\int_{\theta}^{\theta_0} \omega_{k-2} \operatorname{sh} k(\alpha - \theta)\, d\alpha$$
$$u_k = -\frac{1}{k}\omega_k' - \frac{1}{k}\operatorname{sh}\theta\, v_{k-1}$$
$$v_k = -\operatorname{sh}^k\theta \int_0^{\theta_0} \frac{\omega_{k-1}(\alpha)}{\operatorname{sh}^{k+1}\alpha}\, d\alpha - \frac{(k+1)\operatorname{sh}^k\theta}{\operatorname{sh}^{k+1}\theta_0} Q_{k+1}$$
$$p_k = -\frac{1}{k}\int_0^{\theta_0} \omega_{k-2}(\alpha) \operatorname{ch} k\alpha\, d\alpha, \qquad Q_k = \frac{\chi\omega_{k-2}(\theta_0)}{k(k-1)}, \quad \chi = \frac{\beta}{2\lambda h(1-\beta^2)}$$

Analysis of these formulas yields the following estimates:

$$\max_\theta |\omega_{2k}| \leqslant \frac{q_1^k |\omega_0|}{(2^k k!)^2}, \qquad \max_\theta |u_{2k}| \leqslant \frac{q_1^k |\omega_0|}{(2^k k!)^2} \tag{2.5}$$
$$\max_\theta |v_{2k+1}| \leqslant \frac{q_1^k |\omega_0|}{(2^k k!)^2}, \qquad q_1 = 1 + \frac{2\chi}{B}$$

which enable us using the explicit form of functions $\psi(y) \in C^\infty(-\infty, \infty)$ to determine the convergence of series (2.4). In the particular case of bounded of derivatives of $\psi^{(k)}$ and convergence of series

$$\sum_{k=1}^{\infty} \frac{q_1^k}{(2^k k!)^2} \max_y |\psi^{(2k)}|\, r^{2k}$$

then series (2.4) converge absolutely and uniformly when $r < R$ for any R.

3. The motion of a flat piston through a non-homogeneous gas containing dust can be used as a model of expansion of explosion products, and apply the considered above problem to the study of shock wave propagation in coal mines. In the latter coal dust is usually distributed as follows: it is basically concentrated in a thin layer at the wall ($Y = 0$), with some part of it (e.g., dust particles of diameter 0.01—0.25 μm) are suspended. The typical density distribution of coal dust can be approximately defined by the function $\varphi \sim \exp[-(\delta Y)^2]$ (Fig. 1). Hence we shall consider the problem formulated above using function ψ of the form $\psi = \exp(-y^2)$. Solution (2.4) can then be represented in the form

$$\omega = \exp(-y^2)\sum_{k=0}^{\infty} \omega_{2k} H_{2k}(y)\, r^{2k}, \qquad u = \exp(-y^2)\sum_{k=1}^{\infty} u_{2k} H_{2k}(y)\, r^{2k} \tag{3.1}$$
$$v = \exp(-y^2)\left\{[\omega_0 \operatorname{ch}\theta - (A\omega_0 + a)\operatorname{sh}\theta]\, H_1(y)\, r - \sum_{k=1}^{\infty} v_{2k+1} H_{2k+1}(y)\, r^{2k+1}\right\}$$

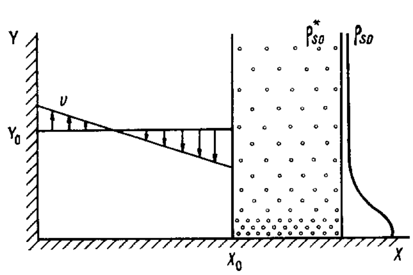

Fig.1

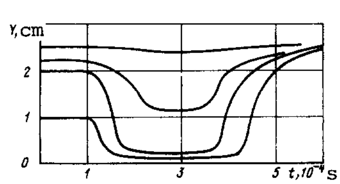

Fig.2

where $H_n(y)$ are Hermite polynomials. Using inequalities (2.5) we can prove that the series in the right-hand sides of (3.1) converge absolutely and uniformly in a finite region and, consequently, formulas (3.1) represent the sought solution.

Let us now consider the initial instant of piston motion which corresponds to small r. Rejecting in (3.1) the terms $O(r^2)$, we obtain

$$\omega = \omega_0 \exp(-y^2), \quad u = 0$$
$$v = [\omega_0 \operatorname{ch} \theta - (A\omega_0 + a) \operatorname{sh} \theta] H_1(y) \exp(-y^2) r$$

Reverting to the old coordinates τ, x, we obtain the following velocity distribution:

$$u = 0, \quad v = 2y[\omega_0\tau - (A\omega_0 + a)x]\exp(-y^2) \tag{3.2}$$

These formulas imply that the particle vertical velocity component

$$v|_{x=\beta\tau} = -2\frac{\beta(\sigma+\alpha)}{\lambda h(\sigma+1)}\tau y \exp(-y^2)$$

is negative at the shock wave front, while the component

$$v|_{x=0} = 2\omega_0\tau y \exp(-y^2), \quad \omega_0 = \beta(1-\sigma)(1-\alpha)(1+\sigma)^{-1}$$

at the piston is positive, since $w_0 > 0$. The latter is due to that

$$\sigma = \frac{1}{M_0^2} < 1, \quad \alpha = \frac{c_s(c_p - c_v)(1+\varkappa_0)(k_0+\sigma)}{k_0(c_v+\varkappa_0 c_s)^2(k_0+1)} \leqslant \frac{c_s(c_p-c_v)}{c_v^2}$$

and the quantity $c_s(c_p - c_v)/c_v^2$ for a mixture of coal dust and air is 0.38.

A characteristic velocity profile is shown in Fig.1 for some in instant of time τ_0.

Thus a (dust) particle immediately after the shock wave passage begins to move downward toward the wall. After some time it stops and, then begins to move upward. On the basis of formulas (3.2) the particle trajectory can be represented as follows:

$$x(\tau) = x_0, \quad \int_{y_0}^{y} \frac{\exp(y^2)}{y} dy = \frac{\beta}{\lambda}[\omega_0(\tau^2 - \tau_0^2) - 2x_0(A\omega_0 + a)(\tau - \tau_0)] \tag{3.3}$$

where x_0, y_0 are the coordinates of the particle at some initial instant of time τ_0. In the coordinate system attached to the piston, particle trajectories are straight lines parallel to the OY axis; the law of motion of particles along these trajectories is determined by the second of formulas (3.3). The curve of function $Y = Y(t)$ defined by that formula is shown in Fig.2 for several Y_0, with $t_0 = 100\ \mu s$ being the instant of time at which the shock wave passes over the particle considered, i.e. $X_0 = Dt_0$ and ($\varkappa_0 = 0.1$, $C = 2\cdot 10^3$ m/s, $\delta = 10^2$ m^{-1}, $\varepsilon = 0.1$, $a_0 = 342.8$ m/s). Within the time interval

$$\Delta\tau = \frac{2\tau_0(\sigma+\alpha)}{\lambda h(1-\sigma)(1-\alpha)}$$

the particle returns to its initial position and, then continues to move upward.

The described effect has a simple explanation. Under otherwise equal conditions, pressure behind the shock wave front is the higher the density ahead of the shock, and the higher the density the lower the shock wave velocity. The equation of the shock wave front is determined by the first and last of Eqs.(1.2), and is of the form

$$g(y, \tau) = -\frac{2\beta(\sigma+\alpha)}{\sigma+1}\tau \exp(-y^2)$$

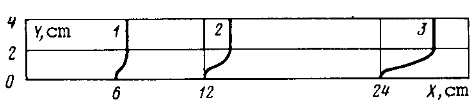

Fig.3

Form of the shock wave is presented in Fig.3 at instants of time $t_1 = 200\ \mu s$, $t_2 = 400\ \mu s$, $t_3 = 800\ \mu s$ for $\varkappa_0 = 0.1$, $C = 2\cdot 10^3$ m/s, $\delta = 10^2$ m^{-1}, $\varepsilon = 0.1$, $a_0 = 342.8$ m/s by curves 1 – 3. It will be seen that the shock wave front is distorted near the wall, by virtue of which particles behind the wave first move down toward the wall and, then because of the pressure gradient in the direction of Y are lifted upward.

An explosion in a mine results in the dust concentrated near the wall becoming suspended after the passage of a shock wave. Experiments had shown the lifting of dust begins after some time interval (e.g., of 300 μs, in experiments described in /7/). The analysis of solution carried out above shows that the effect of time lag in the lifting of dust can be explained within the framework of this problem.

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